A generalization of Scheunert's Theorem on cocycle twisting of color Lie algebras

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Abstract

A classical theorem of Scheunert on G-color Lie algebras, asserts in the case of finitely generated abelian groups, one can twist the algebra structure and the commutation bicharacter on G by a 2-cocycle twist to a super-Lie G graded, algebra. In this paper we show that this can be done for an arbitrary group.

Introduction and notation

We recall first the following definitions (see [Sch] and [Mo]):

Let G be a group and $\chi : G \times G \to k^*$ a bicharacter on G, i.e. a bimultiplicative morphism. We assume that χ is symmetric, i.e. $\chi(h,g)\chi(g,h) = 1$ for all $h, g \in G$. Since k is abelian it follows that χ is trivial for every commutator in G so it factors through $G^{ab} \times G^{ab} \to k^*$. Therefore from now on we assume G to be abelian.

In this paper we assume that k is an algebraically closed field, e will denote the the neutral element of G.

We call L a G-color Lie algebra over k with commutation factor χ if L is a G-graded k-vector space and the bracket $[\ ,\ ] : L \times L \to L$ satisfies:

$[\,a\,,\,b\,] = -\chi(h,g)[\,b\,,\,a\,]$

$\chi(g,k)[\,a\,,[\,b\,,\,c\,]\,] + \chi(k,h)[\,c\,,[\,a\,,\,b\,]\,] + \chi(h,g)[\,b\,,[\,c\,,\,a\,]\,]$ for all $a \in L_g, b \in L_h, c \in L_k$

We say that σ is a 2 cocycle on G if $\sigma : G \times G \to k^*$, satisfies

$\sigma(a,bc)\sigma(b,c) = \sigma(a,b)\sigma(ab,c)$. Then we can define a new bracket $[\ ,\ ]_\sigma : L \times L \to L$ by:

$[\,a\,,\,b\,]_\sigma = \sigma(g,h)[\,a\,,\,b\,]$ for all $a \in L_g, b \in L_h$.

If σ is a 2 cocycle then $\chi_\sigma(g,h) = \chi(g,h)\sigma(g,h)\sigma^{-1}(h,g)$ is a bicharacter. We denote by L^σ the (new) G-color Lie algebra structure on L for this new bracket $[\ ,\ ]_\sigma$ and the commutation factor given by the twisted bicharacter χ_σ.

Let $G_+ = \{g|\chi(g,g) = 1\}$, this is a subgroup of G of index at most 2, we call these the even elements in G. Define the odd elements by: $G_- = \{g|\chi(g,g) = -1\}$, then $G = G_+ \bigcup G_-$.

We may define now $\chi_o(g|h) = 1$ iff at least one of g or h is even else if both G and H are odd $\chi_o(g|h) = -1$

Scheunert's theorem [Sch] shows that for a G color Lie algebra L with bicharacter χ and G a finitely generated abelian group there exists a 2-cocycle σ on G such that the bicharacter $\chi_\sigma = \chi_o$. Thus L^σ can be regarded as a $\mathbb{Z}_2$ graded Lie algebra with the $\mathbb{Z}_2$ (super)bicharacter χ_o.

The proof for an arbitrary abelian group G

In this section we prove:

Theorem *Let G be any abelian group and let L be a G color Lie algebra with commutation factor χ Then there exists a bimultiplicative 2-cocycle σ on G such that the twisted color Lie algebra L^σ is a super-Lie algebra with commutation factor χ_o .*

Proof.
Like in the original proof in [Sche] we may change χ to $\chi\chi_o$ so that we may assume that $\chi(g,g) = 1$ for all $g \in G$. We show then that if $\chi(g,g) = 1$ for all $g \in G$ then there is a cocycle σ on G with $\chi(g,h) = \sigma(g,h)\sigma^{-1}(h,g)$ for all $g,h \in G$. Note that any bimultiplicative map is automatically a 2-cocycle.

To do that we shall use Zorn's lemma. Define a family of subgroups of G
$\mathcal{F} = \{(H,\sigma_H)|H$subgroup of G,$\sigma_H)$ bilinear 2-cocycle on H, $\chi(g,h) = \sigma(g,h)\sigma^{-1}(h,g)\forall g,h \in H\}$

We order this family by $(H',\sigma_{H'}) \preceq (H'',\sigma_{H''})$ iff $H' \subseteq H''$ and $\sigma_{H''}|_{H'} = \sigma_{H'}$. It is clear that $e \in \mathcal{F}$ so that $\mathcal{F}$ is non-void. When in the sequel it is clear on what subgroup σ is defined, we shall not show any more the indices.

This way $\mathcal{F}$ is inductively ordered and by Zorn's lemma there exists a maximal element of $\mathcal{F}$, say (K,σ_K). Assume $K \neq G$. We shall prove this contradicts the maximality of K.

Let t be an element in G that does not belong to K. We look at the subgroup $< t >$ generated by t.

If $< t > \bigcap K = \{e\}$ then let $L =< t > \times K$. Define $\sigma(k,t) = \chi(k,t)$ and $\sigma(t,k) = 1$ for all $k \in K$ and extend σ bimultiplicatively. Since there are no new relations this is well defined and one can see that $\chi(g,h) = \sigma(g,h)\sigma^{-1}(h,g)$ holds on L.

If $< t > \bigcap K =< t^n >$ then there are some $k_1, k_2 \dots k_m \in K$ and some positive integers $n_1, n_2 \dots n_m$ such that $t^n = k_1^{n_1}k_2^{n_2}\dots k_m^{n_m}$. More than one such relations is possible but we just select one, say with a minimal m.

Define now $L =< t, K >$ to be the subgroup generated by K and t. We need to extend σ to L in such a way that :

1) σ is well defined and bimultiplicative on L

2) $\chi(g,h) = \sigma(g,h)\sigma^{-1}(h,g)$ for all $g,h \in L$.

Because $t^n = k_1^{n_1} k_2^{n_2} \dots k_m^{n_m}$ it is clear that for any $u \in L$ we have $\sigma(u,t^n) = \sigma(u, k_1^{n_1} k_2^{n_2} \dots k_m^{n_m})$.

This means is $\sigma(u,t^n)$ is already determined, so loosely speaking we may say $\sigma(u,t) = \sqrt[n]{\prod_i \sigma(u, k_i^{n_i})}$

The problem is that while we have n-th roots, k being algebraically closed, we do not have a uniform *radical function* (say like the real radical), so we need to make sure that we define σ as a function multiplicative on both first and second variable.

We start by defining a multiplicative function $f(u) = \sigma(u,t), f : K \to k^*$ (multiplicative in u) such that:

$f(u)^n = \sigma(u,t^n) = \sigma(u, k_1^{n_1} k_2^{n_2} \dots k_m^{n_m})$ and $f(t^n) = 1$

We let $\mathcal{M}$ be the family of subgroups of K that contain $< t^n >$, on which f can be defined with the above properties, ordered by set inclusion and by the requirement that f extends from the small subgroup to the bigger one.

Then $\mathcal{M}$ is non-void and inductively ordered hence it has a maximal element M. If this maximal element is not K itself say $M \subset K$ and $M \neq K$ then we may contradict the maximality of M.

For an $w \in K - M$ we extend f to $< w, M >$ by:

If $< w > \bigcap M = \{e\}$ then let $f(w)$ be any selection of $\sqrt[n]{\prod_i \sigma(w, k_i^{n_i})}$. This works since $< w, M > = < w > \times M$ and contradicts the maximality of M unless $M = K$.

Else if $< w > \bigcap M = < w^r >$ and $w^r = \prod z_j^{r_j}$, (a finite product) define:

$f(w) = \sqrt[rn]{\prod_{i,j} \sigma(z_j, k_i)^{n_i r_j}}$

This contradicts again the maximality of M and it means there is a multiplicative mapping

$f(u) = \sigma(u,t) : K \to k^*$ such that $f(u)^n = \sigma(u,t^n) = \sigma(u, k_1^{n_1} k_2^{n_2} \dots k_m^{n_m})$ and $f(t^n) = 1$

We use now the required relation to move u on the right side by defining now an analog of f on the "right":

$\sigma(t,u) = \chi(t,u)\sigma(u,t)$

This is multiplicative in the second variable, i.e. in u, because χ is bimultiplicative and also f is multiplicative.

Since $\chi(g,g) = 1$ was granted we define $\sigma(t,t) = 1$, this is consistent with the previous definitions (and this was the reason we asked $f(< t^n >) = 1$).

Now we define σ on all $< t, K >$ by

$$\sigma(t^\alpha u, t^\beta v) = \sigma(t,v)^\alpha \sigma(u,t)^\beta \sigma(u,v)$$

This is bimultiplicative because of the way it was defined. One can use the fact that f is multiplicative, to show that σ is well defined. One needs to show that σ respects the relation: $t^n = k_1^{n_1} k_2^{n_2} \dots k_m^{n_m}$, when substituted on either side. For this we look at reduced forms of $t^\alpha u$, with $\alpha < n$. The relation holds because of the way that $f(t)$ was defined. This way we contradict now the maximality of K so we may conclude $K = G$ and the proof of our theorem.

Remark *There is another really interesting instance of twisting in the paper by Artin-Schelter-Tate [AST]. It is proved there that the multiparametric quantum general linear group is a twist of the standard quantization of the general linear group.*

In fact we are interested in the result of Proposition 1 in [AST], where G is a free abelian group of dimension $n < \infty$, it is shown that any cocycle cohomology class in $H^2(G, k^*)$ contains exactly one bicharacter on G. We conjecture this is the case for an arbitrary abelian group somehow along a construction similar to that of σ in the proof above.

Remark *In fact the proof here does not fully use the fact that k is algebraically closed. Assume that we use transfinite induction to find the following presentation for G: G is given by a system of generators $\{t_\lambda\}_{\lambda\in\Lambda}$ such that for each generator t_λ there is a unique relation $r(t_\lambda) : t^{n_\lambda} = k_1^{n_1} k_2^{n_2} \dots k_m^{n_m}$.*

In our proof we only used the fact that k was closed under radicals of orders equal to the numbers n_λ

Corolaries. The ones given in [Sch]: PBW bases and Ado's Theorem.

References

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